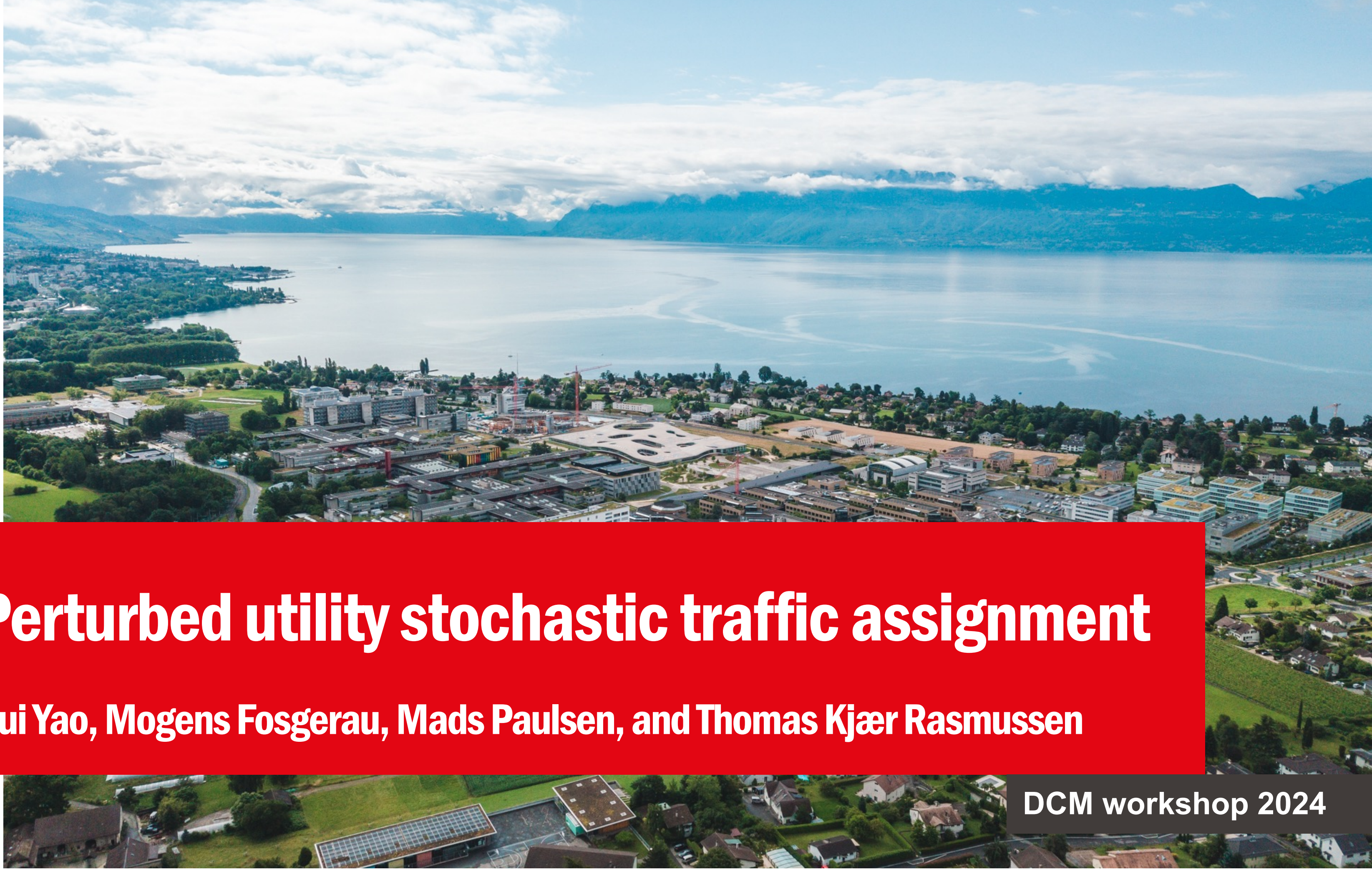

Perturbed utility stochastic traffic assignment

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Outline

- Background
- Perturbed utility SUE
  - Primal formulation
  - Lagrangian dual formulation
  - Quasi-Newton accelerated gradient descent
- Numerical results
- Conclusions

Perturbed utility route choice model (PURC)

- A representative agent w is assumed to solve the following maximization problem in a network $(\boldsymbol{V}, \boldsymbol{E})$:

$$\max_{x^n \in \mathbb{R}_+^{|\boldsymbol{E}|}} \quad u^\top x^w - \boxed{F^w(x^w)} \quad \textbf{Perturbation}$$
$$\text{s.t.} \quad Ax^w = b^w \quad \textbf{Flow conservation}$$

$\boldsymbol{x^w} \in \mathbb{R}_+^{|\boldsymbol{E}|}$ **- link flows**
$u \in \mathbb{R}^{|\boldsymbol{E}|}$ - utility index
$F^w \in \mathbb{R}^{|\boldsymbol{E}|} \to \mathbb{R}$ - convex perturbation function
$A \in \mathbb{R}^{|\boldsymbol{V}| \times |\boldsymbol{E}|}$ - incident matrix
$b^w \in \mathbb{R}^{|\boldsymbol{V}|}$ - unit demand

Key properties

- Allow zero flows on irrelevant links
- Correlation between alternative routes induced directly from network

Perturbed utility route choice model (PURC)

$$\min_{x^n \in \mathbb{R}_+^{|E|}} \quad c^\top x^w + F^w(x^w)$$
$$\text{s.t.} \quad Ax^w = b^w \qquad (\eta)$$

- *Assumptions: network perturbation function*

$$F^w(x) = \sum_{ij} F_{ij}^w\left(x_{ij}^w\right)$$

  - F_{ij}^w *is link-specific, continuous differentiable, strictly convex, and strictly increasing*
  - $F_{ij}^w(0) = F_{ij}^{w\prime}(0) = 0$
  - c *is a vector of positive link cost*

Perturbed utility route choice model (PURC)

- Complementarity condition:

$$0 \leq x_{ij}^{w*} \perp \left[c_{ij} + F_{ij}^{w\prime}\left(x_{ij}^{w*}\right) + \eta_j^{w*} - \eta_i^{w*}\right] \geq 0$$

- Estimation – Given x_{ij}^{w*}, estimate parameters β in c_{ij}
- **How to predict?** – Given β, solve for link flows x_{ij}^{w*}

For any given $c_{ij} > 0$,

$$x_{ij}^{w*} = \left(F_{ij}^{w\prime}\right)^{-1}\left(\eta_i^{w*} - \eta_j^{w*} - c_{ij}\right)$$

Exploiting the FOC
(Key ingredient)

- **What if cost is flow-dependent $c_{ij}(x_{ij})$?**

Perturbed utility stochastic traffic assignment
Primal formulation

- **Primal formulation – Constrained optimization**
  For a set of traveler types $\{w\}$, and demands $\{q^w\}$

$$\min_{x\geq 0} Z = \sum_{ij} \sum_{w} \left[\int_0^{\sum_{w'} q^{w'} x_{ij}^{w'}} t_{ij}(m)dm + q^w F_{ij}^w\left(x_{ij}^w\right) \right]$$

Beckmann's UE equation **Perturbation**

$$s.t. \qquad Ax^w - b^w = 0, \forall\, w$$

Flow conservation

- ***Resulting SUE (optimal condition) is equivalent to FOC of PURC***

Assumption: t_{ij} is positive, differentiable, increasing and strictly convex

Perturbed utility stochastic traffic assignment

Lagrangian dual formulation

Recall $x_{ij}^{n*} = \left(F_{ij}^{n\prime}\right)^{-1}\left(\eta_i^{n*} - \eta_j^{n*} - c_{ij}^{*}\right)$

With flow-dependent $c_{ij}^{*} = t_{ij}\left(\sum_{w'} q^{w} x_{ij}^{w*}\right)$

- **Lagrangian dual formulation – Unconstrained optimization!**

$$\max_{\eta} G = \sum_{ij}\sum_{w}\left[\int_0^{\sum_{w'} q^{w'} x_{ij}^{w'*}} t_{ij}(m)dm + q^{w} F_{ij}^{w}\left(x_{ij}^{w*}\right)\right] - \sum_{w} \eta^{w}\left(Ax^{w} - b^{w}\right)$$

Lemma (Strong duality)

The duality gap between the primal TAP problem and the corresponding dual problem at their optimal solutions is zero.

Rui Yao

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Quasi-Newton accelerated gradient descent

$$x_{ij}^{w*} = \left(F_{ij}^{w'}\right)^{-1}\left(\eta_i^{w*} - \eta_j^{w*} - c_{ij}^{*}\right) \quad \Longleftrightarrow \quad c_{ij}^{*} = t_{ij}\left(\sum_{w'} q^w x_{ij}^{w*}\right)$$

Interdependent

- **Iterative update x_{ij}^{n*} with estimates of c_{ij}^{*} (Partial linearization)**
- **Update estimates of c_{ij}^{*} by solving an auxiliary fixed point**

$$U_{ij}\left(x_{ij}^{w*}, c_{ij}^{*}\right) = t_{ij}\left(\sum_{w'} q^w x_{ij}^{w*}\right) - c_{ij}^{*} = 0$$

- **At each iteration, update one Newton-step of the auxiliary fixed point**

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Quasi-Newton accelerated gradient descent

$$\max_{\eta} G = \sum_{ij}\sum_{w}\left[\int_0^{\sum_{w'} q^{w'} x_{ij}^{w'*}} t_{ij}(m)dm + q^w F_{ij}^w\left(x_{ij}^{w*}\right)\right] - \sum_{w} \eta^w (Ax^w - b^w)$$

$$\Delta = \frac{\dfrac{\partial G}{\partial \eta_i^w} = q^w (A_i x^w - b^w) \quad \textbf{Gradient}}{\dfrac{\partial^2 G}{\partial^2 \eta_i^w} = q^w A_i \nabla_{\eta_i^w}\left(F_{ij}^{w'}\right)^{-1} \quad \textbf{Hessian Diagonal}}$$

Quasi-Newton

$$\tilde{\eta}_j^{w(m+1)} = \eta_j^{w(m)} + \gamma \Delta^{(m)}$$

$$\eta_j^{w(m+1)} = \tilde{\eta}_j^{w(m+1)} + \frac{m}{m+\alpha}\left(\tilde{\eta}_j^{w(m+1)} - \tilde{\eta}_j^{w(m)}\right)$$

Nesterov's momentum acceleration

Perturbed utility stochastic traffic assignment

Quasi-Newton accelerated gradient descent

- For each iteration:
  1. PURC assignment

$$x_{ij}^{w*(m+1)} = \left(F_{ij}^{n\prime}\right)^{-1} \left(\eta_i^{n*(m)} - \eta_j^{n*(m)} - c_{ij}^{*(m)}\right)$$

  2. Update dual variables with qN-AGD*

$$\tilde{\eta}_j^{w(m+1)} = \eta_j^{w(m)} + \gamma_1 \Delta^{(m)}$$

$$\eta_j^{w(m+1)} = \tilde{\eta}_j^{w(m+1)} + \frac{m}{m+\alpha}\left(\tilde{\eta}_j^{w(m+1)} - \tilde{\eta}_j^{w(m)}\right)$$

  3. Update link costs with one Newton-step

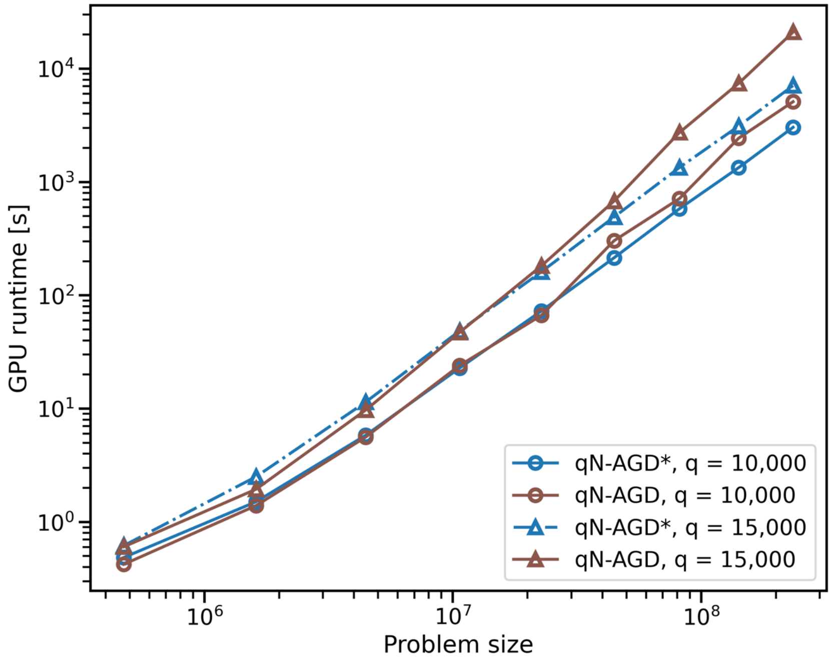
$$c_{ij}^{*(m+1)} = c_{ij}^{*(m)} - \gamma_1 \frac{U_{ij}\left(x_{ij}^{w*(m+1)}, c_{ij}^{*(m)}\right)}{\nabla_{c_{ij}^*} U_{ij}\left(x_{ij}^{w*(m+1)}, c_{ij}^{*(m)}\right)}$$

Quasi-Newton dual algorithm performance

Dual algorithm runtime performance
(comparison with existing algorithms in the literature)

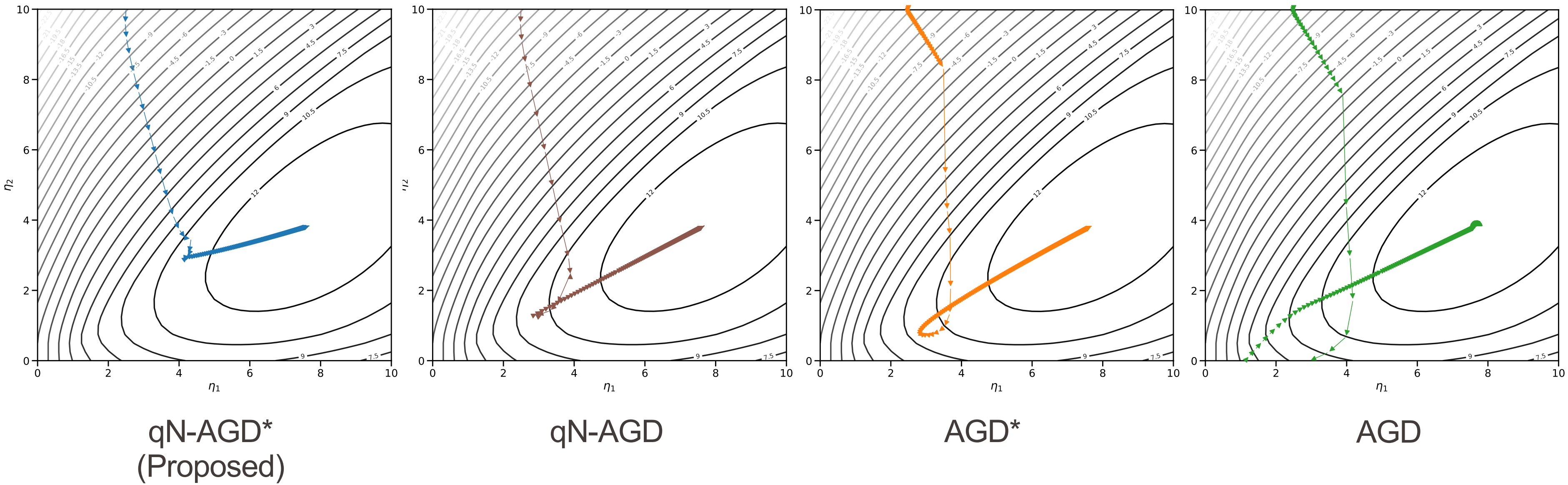
Network	Problem size	Runtime [s]			
	|N| x |W|	**Proposed**	qN-AGD	AGD*	AGD
Sioux Falls	1.27E+04	**0.25**	0.43	1.95	5.78
Berlin-Friedrichshain	1.13E+05	**2.17**	7.23	42.83	144.12
Berlin-Tiergarten	2.31E+05	**3.23**	7.70	142.85	410.33
Anaheim	5.85E+05	**0.58**	0.65	16.87	26.71
Berlin-Center	9.26E+06	**68.99**	72.01	1487.23	3742.31
Chicago-Sketch	8.69E+07	**94.90**	125.77	7196.18	9322.30

Quasi-Newton dual algorithm performance

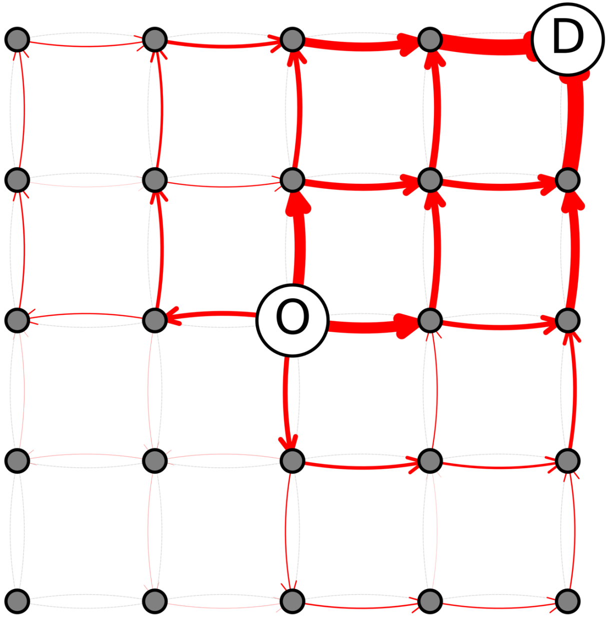


Quasi-Newton dual algorithm performance

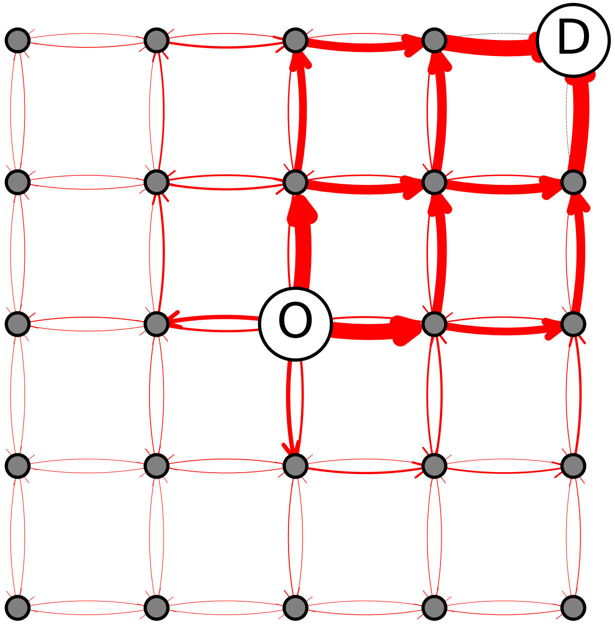
Solution trajectory



PURC vs NGEV SUE



PURC
(perturbation scale $\mu = 10$)



NGEV
(proportionality parameter = 10)

Conclusions

- Main takeaways
  - Equivalent unconstrained Lagrangian dual formulation
    for PURC SUE with flow-dependent costs
  - Fast assignment algorithm with potential for very large application
  - Predicted equilibrium pattern is plausible
  - Applicable for flow-independent problems,
    typical setting for choice model prediction.
- Future directions
  - Application in design problems
  - Modeling and prediction in other (virtual) networks?

EPFL

Thanks!

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