Predictive Error Bounds for the Random Parameters Logit Assortment Problem

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The Problem

candidate products i	1	2	opt-out
"revenue" r_i	2	4	0
assortment $S = 1$ (availability)	1	0	1
assortment $S = 2$ (availability)	0	1	1
assortment $S = 3$ (availability)	1	1	1
demand (probabilities) for $S = 1$	.5	0	.5
demand (probabilities) for $S = 2$	0	.5	.5
demand (probabilities) for $S = 3$	.33	.33	.33
expected revenue (objective) for $S = 1$	$2 \cdot 0.5 + 4 \cdot 0 = 1$		
expected revenue (objective) for $S = 2$	$2 \cdot 0 + 4 \cdot 0.5 = 2$		
expected revenue (objective) for $S = 3$	$2 \cdot 0.33 + 4 \cdot 0.33 = 1.98$		

(may be an) NP-hard problem (2^N feasible solutions) - but very relevant in industry

Mixed-Logit Assortment

Mixed-Logit Assortment (Rusmevichientong et al. 2014)

- $N := 1, \ldots, n$ candidate products
- v_i is ”random” in MXL
  - Our case $v_i = -\beta \cdot r_i \quad \rightarrow \beta$ follows continuous distribution
  - $v_{\text{opt-out}} = 0$, thus $e^{v_{\text{opt-out}}} = 1$
- Select a subset of N to be offered to customers that maximizes expected revenue

$$Z^* := \max \sum_{i \in N} r_i \cdot \underbrace{E\left[\frac{e^{v_i} Y_i}{1 + \sum_{j \in N} e^{v_j} Y_j}\right]}_{P_i^{\text{MXL}}} \tag{1}$$

$$\text{s.t.} \quad Y_i \in \{0,1\} \, \forall i \in N \quad \text{decision variable} \rightarrow \text{assortment} \tag{2}$$

Simulation-Based Approximation of P_i^{MXL}

$$P_i = \int L_i(\beta) \cdot f(\beta) d\beta = \int L_i(\beta) dF(\beta) = E\left[\frac{e^{v_i} Y_i}{1 + \sum_{j \in N} e^{v_j} Y_j}\right]$$

P_i is an indefinite integral form $\rightarrow$ no analytical solution

$\Rightarrow$ approximate using Monte-Carlo Simulation

$$\hat{P}_i = \frac{1}{G} \sum_{g=1}^{G} L_i^g(\beta^g) = \frac{1}{G} \sum_{g=1}^{G} \underbrace{\left(\frac{e^{-\beta^g r_i} Y_i}{1 + \sum_{j \in N} e^{-\beta^g r_j} Y_j}\right)}_{L_i^g(\beta^g)} \qquad (3)$$

- G number of draws (scenarios, realizations, samples)
- $G \rightarrow +\infty$, then $\hat{P}_i = P_i$

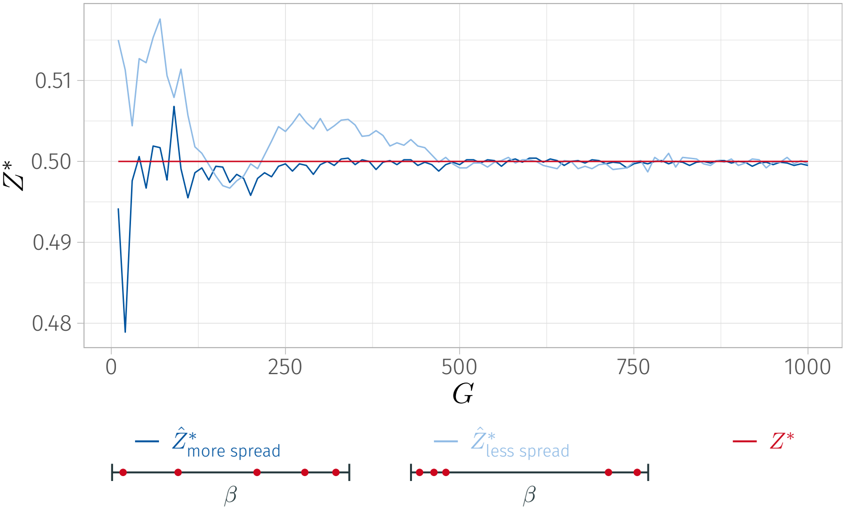
MXL-ASSORT aka LC-ASSORT

$$\hat{Z}^* := \max \sum_{i \in N} r_i \cdot \underbrace{\frac{1}{G} \sum_{g=1}^{G} \left(\frac{e^{-\beta^g r_i} Y_i}{1 + \sum_{j \in N} e^{-\beta^g r_j} Y_j} \right)}_{\hat{P}_i} \tag{4}$$

$$\text{s.t.} \quad Y_i \in \{0, 1\}, \forall i \in N \tag{5}$$

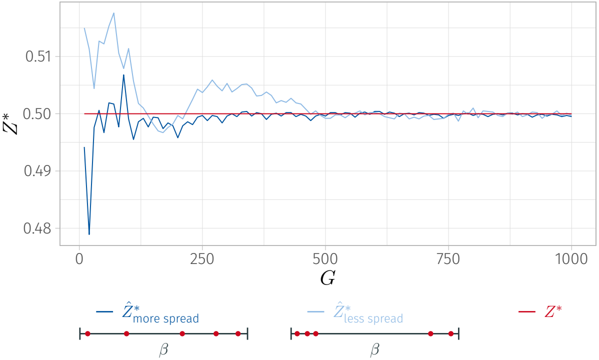
- Optimization problem of dimension $N \times G$
- NP-complete with $G \geq 2$
- If $G \to +\infty$, then $\hat{Z}^* = Z^*$
- Concavity proof (Benati and Hansen 2002) does not hold because of r_i
- We use MIP reformulation based on Haase 2009 and Méndez-Díaz et al. 2014

Mixed-Logit Assortment (Approximation Quality vs. G)



Mixed-Logit Assortment (Approximation Quality vs. G)

- Larger G requires more computational resources
- $\hat{P}_i$ and hence $\hat{Z}^*$ is more accurate if draws are spread out (Train 2009)
- Question: How do we measure the approximation quality $\hat{Z}^*$?



Why should I care?

I want to know a priori how trustworthy is my solution (objective function value) given a certain ”cost” (run-time)?

- MXL and its parameters are given (estimated)
- Solving MXL-ASSORT (optimization problem): trade-off between run-time and draws G (run-time grows exponentially in G) versus approximation quality ($\hat{Z}^*$ vs Z^*)
- Before solving the problem, I want to know how close I can get to the red line when I ”invest” G draws?

Approximation Quality

Approximation Quality

Quality of $\hat{P}_i$ is dependent on:

- Range of possible β values

G Number of draws

VR Variance reduction techniques (drawing strategies)

Koksma-Hlawka Inequality (Niederreiter 1992; Pausinger and Svane 2015):

$$\left| \underbrace{\frac{1}{G}\sum_{g=1}^{G} L_i\left(\beta^g\right)}_{\text{approximation } (\hat{P}_i)} - \underbrace{\int L_i\left(\beta\right) dF(\beta)}_{\text{actual } (P_i)} \right| \leq \underbrace{\underbrace{V\left(L_i\right)}_{\text{variation}} \cdot \underbrace{D_G\left(\beta_1, \ldots, \beta_G\right)}_{\text{discrepancy}}}_{\text{upper bound}} \tag{6}$$

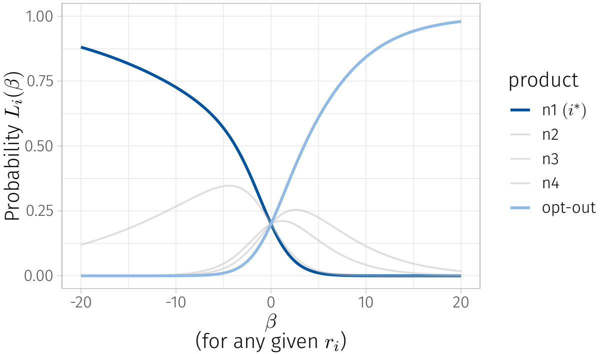
$$L_i(\beta) = \frac{e^{-\beta r_i} Y_i}{1 + \sum_{j \in N} e^{-\beta r_j} Y_j}$$

L_i monotonic w.r.t β for:

- opt-out alternative
- product with the highest utility U

$$V(L_{\text{opt-out}}) = L_{\text{opt-out}}(b) - L_{\text{opt-out}}(a)$$
$$V(L_{i^*}) = L_{i^*}(a) - L_{i^*}(b)$$

i^* is the product with the highest price

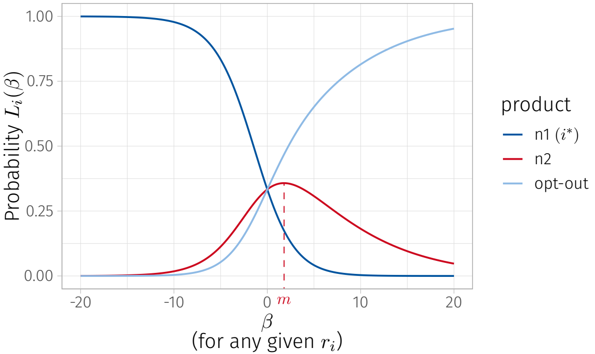


Exactly one maximum m

$$V(L_i) = |L_i(m) - L_i(a)| + |L_i(b) - L_i(m)|$$
$$= 2L_i(m) - L_i(a) - L_i(b)$$

if $m \in [a, b]$, otherwise

$$V(L_i) = |L_i(a) - L_i(b)|$$

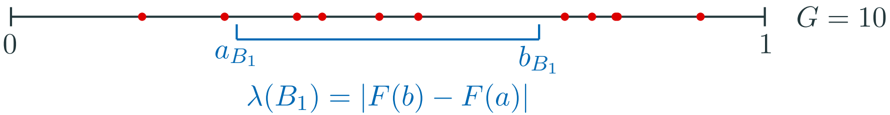


Question 1

How to prove that the MNL choice probability for a given alternative has at most one extreme point?

$D_G(\beta_1, \ldots, \beta_G)$ is the **discrepancy** for the sequence of draws $(\beta_1, \ldots, \beta_G)$

$$D_G(\beta_1, \ldots, \beta_G) = \sup_{B \in \mathbb{B}} \left| \frac{\mathbb{I}_B(\beta_g)}{G} - \lambda(B) \right| \tag{7}$$

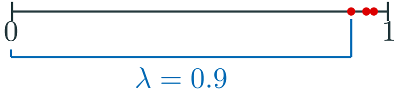


- Discrepancy measures the "quality" for the sequence of draws $(\beta_1, \ldots, \beta_G)$
- $\mathbb{I}_B$ is a counting function for number of $\beta_g \in B$
- $\lambda(B)$ is the Lebesgue measure of B $(\lambda(B) = |F(b) - F(a)|)$
- Dependent on the number of draws G and the drawing strategies VR

- $G = 3$ Standard Uniform $U(0, 1)$
- $0.33 \leq D_G \leq 1$
- Example: $P(D_G \leq 0.9)$

Case 1: large interval, low number of draws

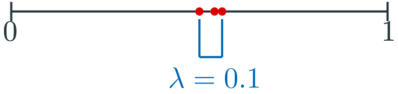
$B^* := 0$ draws with $\lambda = 0.9$



$D_G = 0.9$

Case 2: small interval, high number of draws

$B^* := 3$ draws with $\lambda = 0.1$



$D_G = 0.9$

Probabilistic Discrepancy: $P\left(D_G \leq d\right)$

- Add probabilities of scenarios by using order statistics $\left(\beta_{(1)} \leq \beta_{(2)} \leq \ldots \leq \beta_{(G)}\right)$
- Use multidimensional integration to calculate the $P\left(D_G \leq d\right)$
- Only needed for certain drawing strategies
  - Pseudo-Random
  - Systematic Sampling
  - Antithetics
- For Halton-Sequences and Systematic Number, Discrepancy is trivial

Goal: small G, small d, and large $P\left(D_G \leq d\right)$.

Question 2

What do I lose in prediction when not using pseudo-random numbers? Why should I care?

First Numerical Experiments

Settings

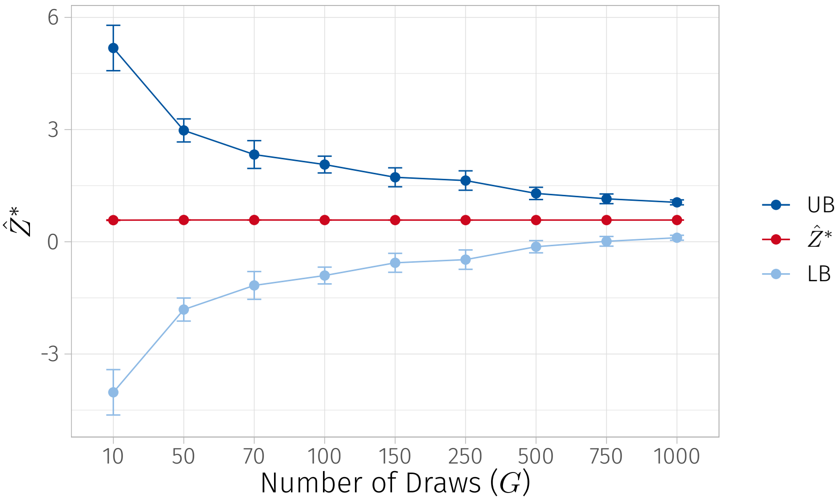
- r_i is randomly generated ($r_i \sim U[0,1]$)
- r_i values are constant for each scenario.
- Set of Scenarios:
  - Number of Products (N): 10
  - Number of Draws (G): $\{10, 50, 70, 100, 150, 250, 500, 750, 1000\}$ with $\beta \sim U[0,1]$
  - Drawing Strategies: Pseudo-Random Numbers (PRN), Antithetics (ANT), and Halton Sequence

Results for Pseudo-Random-Numbers (PRN)

G	$\hat{Z}^*$	Variation $\sum_{i \in n} r_i \cdot 2$	Discrepancy (D_G)	Upper Bound of $\lvert Z^* - \hat{Z}^* \rvert$
10	0.5781	11.2595	0.4088	4.6026
50	0.5828	11.2595	0.2127	2.3948
70	0.5827	11.2595	0.1554	1.7498
100	0.5813	11.2595	0.1317	1.4834
150	0.5811	11.2595	0.1015	1.1432
250	0.5803	11.2595	0.0939	1.0571
500	0.5807	11.2595	0.0633	0.7125
750	0.5803	11.2595	0.0504	0.5678
1,000	0.5812	11.2595	0.0420	0.4732

Example for G = 1,000: Variation × Discrepancy, i.e., 11.26 × 0.0420 = 0.4732
$\Rightarrow 0.5812 - 0.4732 \leq Z^* \leq 0.5812 + 0.4732$

Experimental Results: PRN



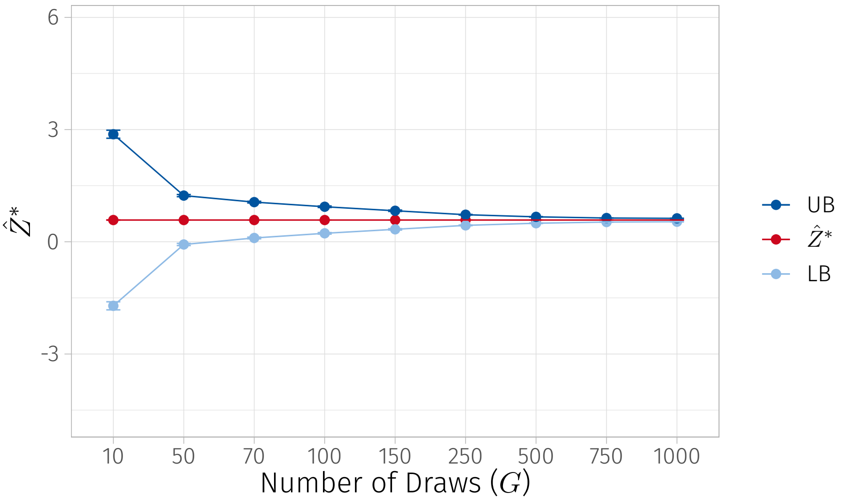
Results for Halton Sequence

G	$\hat{Z}^*$	Variation $\sum_{i\in n} r_i \cdot 2$	Discrepancy (D_G)	Upper Bound of $\lvert Z^* - \hat{Z}^*\rvert$
10	0.5817	11.2595	0.2037	2.2936
50	0.5815	11.2595	0.0580	0.6534
70	0.5814	11.2595	0.0425	0.4793
100	0.5815	11.2595	0.0315	0.3552
150	0.5814	11.2595	0.0220	0.2483
250	0.5814	11.2595	0.0127	0.1424
500	0.5814	11.2595	0.0076	0.0856
750	0.5814	11.2595	0.0048	0.0537
1,000	0.5814	11.2595	0.0040	0.0452

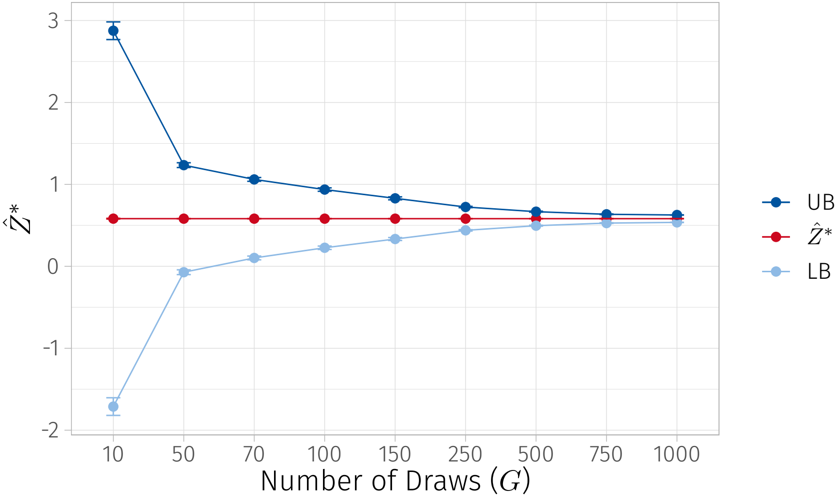
Example for G = 1,000: Variation × Discrepancy, i.e., 11.26 × 0.0420 = 0.4732

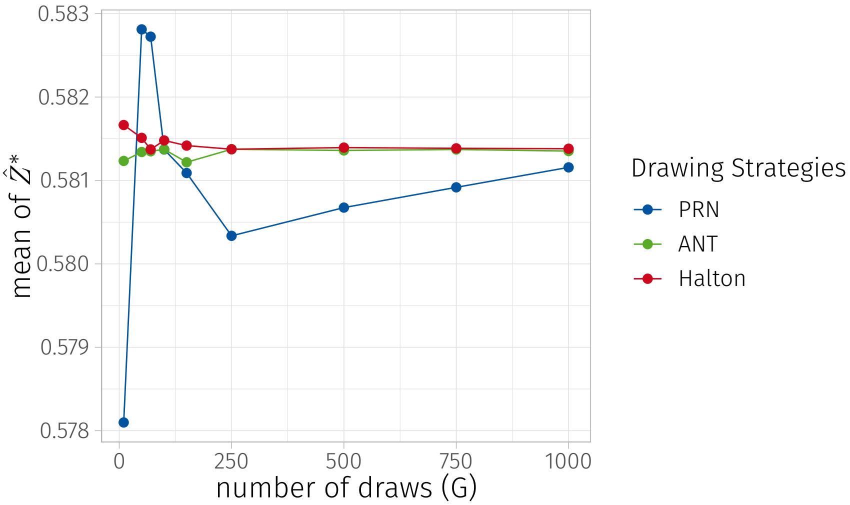
$$\Rightarrow 0.5812 - 0.4732 \leq Z^* \leq 0.5812 + 0.4732$$

Experimental Results: Halton

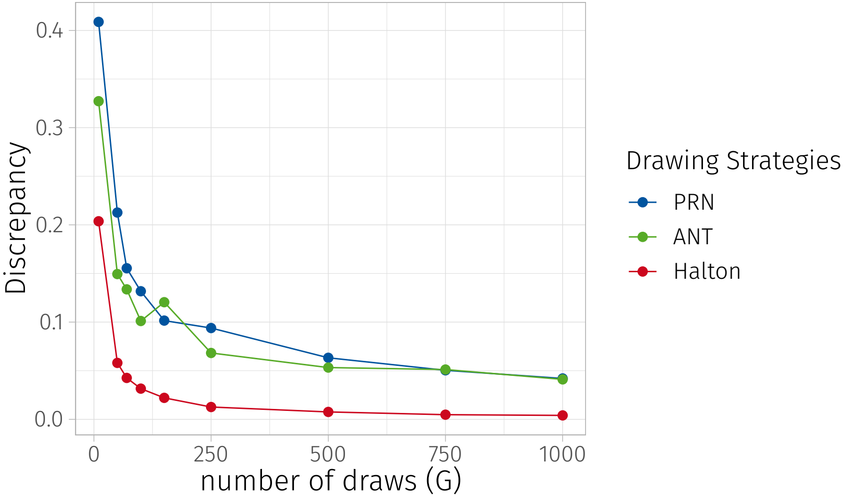


Experimental Results: Halton (Enlarged)




0.583
0.582
0.581
0.580
0.579
0.578
mean of $\hat{Z}^*$
0
250
500
750
1000
number of draws (G)
Drawing Strategies
PRN
ANT
Halton

Experimental Results



Conclusion

Take home for you

- Quality of $\hat{P}_i$ is dependent on the range of β, number of draws G, and VR techniques.
- Trade-Off in G : approximation quality vs. computational resources
- Approximation guarantee (upper bound) supports you in selecting appropriate G
- Our results guide you in accelerating your computational studies

Take home for me

Q1 MNL probability function has at most one extreme point (variation)

Q2 MXL prediction using non-random numbers: do I lose important properties? (variance reduction techniques)

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Appendix

Small Experiment: Settings

2-Product MXL Assortment Problem

- Variation $V(L_i(\beta))$ can be obtained in this setting
- Set of Scenarios:
  - Number of Products $(N) = 2$
  - Produce Price $\{r_1, r_2\} = \{1, 5\}$
  - Number of Draws $(G) = \{10, 100, 500, 1000\}$ with $\beta \sim U\,[0, 1]$
  - Drawing Strategies: Pseudo-Random Numbers (PRN), Antithetics (ANT)[1]

Upper Bound for $|Z^* - Z|$:

$$|Z^* - Z| \leq \left(\sum_{i \in n} r_i \cdot V(L_i(\beta)) \right) \cdot D_G\left(\beta_1, \ldots, \beta_G\right)$$

[1] The half of other draws are mirror images of the other half

Small Experiment: Results

VR	Z^*	Variation ($\sum_{i \in n} r_i \cdot V(L_i(\beta))$)	Discrepancy (D_G)	Upper Bound of $\lvert Z^* - Z \rvert$	G
PRN	0.896	1.796	0.404	0.726	10
PRN	0.845	1.796	0.155	0.278	100
PRN	0.812	1.796	0.092	0.165	500
PRN	0.759	1.796	0.058	0.104	1000
ANT	0.817	1.796	0.350	0.629	10
ANT	0.783	1.796	0.142	0.255	100
ANT	0.783	1.796	0.046	0.083	500
ANT	0.773	1.796	0.034	0.061	1000

Approximation Quality (Variation)

$V(L_i)$ is the **bounded variaton** of the logit probability function

$$V(L_i) = \sup_{B \in \mathbb{B}} \left\{ \sum_{k=0}^{K-1} |L_i(\zeta_{k+1}) - L_i(\zeta_k)| \right\} \tag{8}$$

- $V(L_i)$ measures the changes in L_i as β varies.
- ζ_k is a break point where $\zeta_k \in [a, b]$

  $\rightarrow$ $a, b \in \mathbb{R}$ and within the range β
- $B = \{\zeta_0, \dots, \zeta_K\}$ is a partition of $\mathbb{B} := [a, b]$
- K : number of break points in each B
- Idea: Find the partition B s.t. $V(L_i)$ is the largest

Issue with Variation (2)

Even if issue (1) is true...

$$L_i(\beta) = \left(\frac{e^{-\beta r_i} Y_i}{1 + \sum_{j \in N} e^{-\beta r_j} Y_j} \right)$$

find β s.t. $\frac{dL_i(\beta)}{d\beta} = 0$

$$\frac{dL_i(\beta)}{d\beta} = \left(\frac{(-r_i e^{-\beta r_i} Y_i)(1 + \sum_{j \in N} e^{-\beta r_j} Y_j) - (\sum_{j \in N} -r_j e^{-\beta r_j} Y_j)(e^{-\beta r_i} Y_i)}{\left(1 + \sum_{j \in N} e^{-\beta r_j} Y_j\right)^2} \right) = 0$$

$$= -r_i + \sum_{j \in N} (r_j - r_i) e^{-\beta r_j} Y_j = 0$$

L_i is still dependent on the assortment

Another approximation method is needed (Newton's method) to solve for β

Assumption for Variation

If one unique maximum m exists when $n > 2$, then for each alternative:

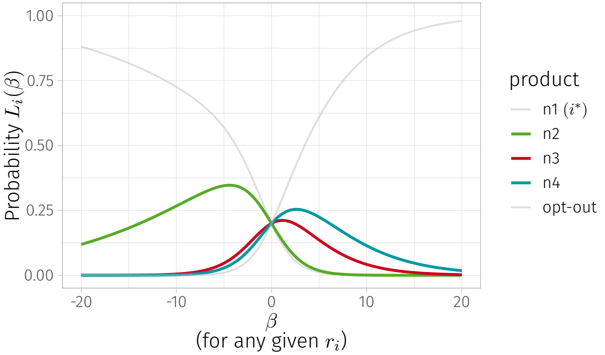
- $2 \cdot L_i(m)$ is the upper bound for $V(L_i)$
- The global upper bound (regardless of assortment) is 2

Assumption for Variation: Getting Global Upper Bound

Assumption: one unique maximum m exists when $n > 2$

Getting global upper bound:

- for products outside i^*: $L_i(a) = L_i(b) = 0$
- thus, $V(L_i) = 2 \cdot L_i(m)$
- since L_i is a probability function, thus its maximum value is 2.
- Therefore, maximum value for $V(L_i) = 2$



Types of Drawings

Type of drawing methods:

- Pseudorandom Numbers
- Antithetics
- Systematic Draws (draw at each interval)
- Systematic Draws (draw at first interval)
- Systematic Numbers
- Halton Sequence

Discrepancy (Worst-Case Scenario)

Find the **worst-case scenario** from all possible S for a given VR technique

- Case 1: All draws are arbitrarily close to each other

- Case 2: All draws are arbitrarily close to 0 or 1

- In both cases, $D_G \to 1$
- The chance of the worst-case scenarios is low

Idea: Find $P\left(D_G \leq d\right)$

Discrepancy (Probabilistic): Example

$G = 3$ (Pseudo-random Numbers), Standard Uniform $U(0, 1)$

Best Case Scenario

draws are spread evenly

$D_G = 0.33$

Worst Case Scenario

draws are arbitrarily close to each other

$D_G \to 1$

Thus, D_G must be between 0.33 and 1.

Discrepancy (Probabilistic): Back to Example

- $G = 3$ (Pseudo-random Numbers), Standard Uniform $U(0,1)$
- $0.33 \leq D_G \leq 1$
- Order Statistics: $\left(\beta_{(1)} \leq \beta_{(2)} \leq \beta_{(3)}\right)$
- Joint density of Uniform Order Statistic: $f_{\beta_{(1)},\beta_{(2)},\dots,\beta_{(G)}}(\beta_1, \beta_2, \dots, \beta_G) = G!$
- Example: $P\left(D_G \leq 0.9\right)$
- Iteratively determine the range of each draw s.t. $D_G \leq 0.9$.

$\rightarrow\ \beta_{(1)} \in [0, 0.9]$
$\rightarrow\ \beta_{(2)} \in [\beta_{(1)}, \min(\beta_{(1)} + 0.9, 1)]$
$\rightarrow\ \beta_{(3)} \in [\max(\beta_{(1)} + 0.1, \beta_{(2)}), \min(\beta_{(2)} + 0.9, 1)]$

$$
\begin{aligned}
P\left(D_G \leq 0.9\right) = & P(0 \leq \beta_{(1)} \leq 0.9) \times \\
& P\left(\beta_{(1)} \leq \beta_{(2)} \leq \min(\beta_{(1)} + 0.9, 1)\right) \times \\
& P\left(\max(\beta_{(1)} + 0.1, \beta_{(2)}) \leq \beta_{(3)} \leq \min(\beta_{(2)} + 0.9, 1)\right)
\end{aligned}
$$

Discrepancy: Pseduorandom and Antitethics

Let $P := [c, d]$ be a subinterval of $[0, 1]$.

Pseudorandom Numbers

$D_G \to 1$

Antithetics

$D_G \to 1$

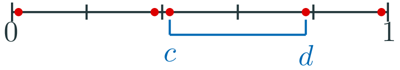
Discrepancy: Systematic Draws

We divide $[0, 1]$ into Q segments with equal length.

- Systematic Draws (each): take a draw at each segment Q
- Systematic Draws (first): take a draw at the first segment only, the rest are added s.t. each draw has an equal distance.

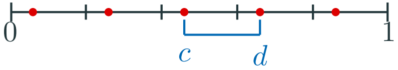
Let $P := [c, d]$ be a subinterval of $[0, 1]$ and $S = 5$.

Systmatic Draws (each)



$D_G \to 2/Q$

Systmatic Draws (first)

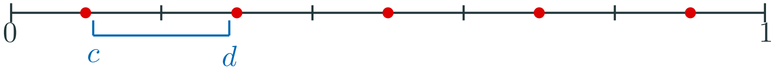


$D_G \to 1/Q$

Discrepancy: Systematic Numbers and Halton Sequence

Let $P := [c, d]$ be a subinterval of $[0, 1]$ and $Q = 5$.

Systematic Numbers



Since the "draw" point is always the midpoint of each interval, thus $D_G \to 1/Q$.

Halton Sequence (Niederreiter 1992),

$$D_G < 2\left(\frac{1}{G} + \frac{1}{G}\left(\frac{b_i - 1}{2\log b_i}\log G + \frac{b_i + 1}{2}\right)\right) \tag{9}$$

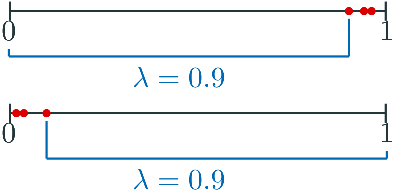
where b_i is the prime base for the Halton sequence.

Discrepancy (Probabilistic): Example

Same Case, different Scenarios:

Case 1: large interval, low number of draws

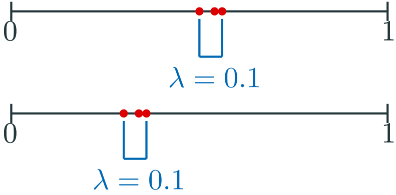
$S^* := 0$ draws with $\lambda = 0.9$



$D_G = 0.9$

Case 2: small interval, high number of draws

$S^* := 3$ draws with $\lambda = 0.1$



$D_G = 0.9$

Discrepancy (Probabilistic): Obtaining Range for Draws

For each pairing $(\beta_{(i)}, \beta_{(j)})$ where $i > j$:

1. Create segment $S_{\beta_{(i)},\beta_{(j)}}$ and determine its the range.

$$\left(\pm d + \frac{\mathbb{I}^{\text{inclusive}}_{S_{\beta_{(i)},\beta_{(j)}}}(\beta_{(i)}, \beta_{(j)})}{G} \right) \leq \lambda(S_{\beta_{(i)},\beta_{(j)}}) \leq \left(\pm d + \frac{\mathbb{I}^{\text{exclusive}}_{S_{\beta_{(i)},\beta_{(j)}}}(\beta_{(i)}, \beta_{(j)})}{G} \right)$$

$$\gamma^{\text{min}}_{S_{\beta_{(i)},\beta_{(j)}}} \leq \lambda(S_{\beta_{(i)},\beta_{(j)}}) \leq \gamma^{\text{max}}_{S_{\beta_{(i)},\beta_{(j)}}}$$

Discrepancy (Probabilistic): Obtaining Range for Draws

$$\left(\pm d + \frac{\mathbb{I}^{\text{inclusive}}_{S_{\beta_{(i)},\beta_{(j)}}}\left(\beta_{(i)},\beta_{(j)}\right)}{G}\right) \leq \lambda(S_{\beta_{(i)},\beta_{(j)}}) \leq \left(\pm d + \frac{\mathbb{I}^{\text{exclusive}}_{S_{\beta_{(i)},\beta_{(j)}}}\left(\beta_{(i)},\beta_{(j)}\right)}{G}\right)$$

$$\gamma^{\text{min}}_{S_{\beta_{(i)},\beta_{(j)}}} \leq \lambda(S_{\beta_{(i)},\beta_{(j)}}) \leq \gamma^{\text{max}}_{S_{\beta_{(i)},\beta_{(j)}}}$$

- $\beta_{(1)}$ must be lower than $F^{-1}(d)$
- $0 \leq \gamma^{\text{min}}_{S_{\beta_{(i)},\beta_{(j)}}} \leq \lambda(S_{\beta_{(i)},\beta_{(j)}}) \leq \gamma^{\text{max}}_{S_{\beta_{(i)},\beta_{(j)}}} \leq 1$
- Take lower value for $\gamma^{\text{min}}_{S_{\beta_{(i)},\beta_{(j)}}}$ and higher value for $\gamma^{\text{max}}_{S_{\beta_{(i)},\beta_{(j)}}}$
- If no valid values, then the pairing have no dependent min/max range.

Discrepancy (Probabilistic): Obtaining Range for Draws

Order Statistics: $\beta_{(1)} \leq \beta_{(2)} \leq \beta_{(3)}$

Pairing $(\beta_{(1)}, \beta_{(2)})$

$$\gamma^{\text{min}}_{S_{\beta_{(1)},\beta_{(2)}}} = \pm d + \frac{\mathbb{I}^{\text{inclusive}}_S(\beta_g)}{G} = \pm 0.9 + \frac{2}{3} = 1.57 \text{ or } -0.23$$

$$\gamma^{\text{max}}_{S_{\beta_{(1)},\beta_{(2)}}} = \pm d + \frac{\mathbb{I}^{\text{exclusive}}_S(\beta_g)}{G} = \pm 0.9 + \frac{0}{3} = 0.9 \text{ or } -0.9$$

- No minimum gap between $\beta_{(1)}$ and $\beta_{(2)}$
- Maximum gap between $\beta_{(1)}$ and $\beta_{(2)}$ is 0.9

Pairing $(\beta_{(2)}, \beta_{(3)})$

$$\gamma^{\text{min}}_{S_{\beta_{(2)},\beta_{(3)}}} = \pm d + \frac{\mathbb{I}^{\text{inclusive}}_S(\beta_g)}{G} = \pm 0.9 + \frac{2}{3} = 1.57 \text{ or } -0.23$$

$$\gamma^{\text{max}}_{S_{\beta_{(2)},\beta_{(3)}}} = \pm d + \frac{\mathbb{I}^{\text{exclusive}}_S(\beta_g)}{G} = \pm 0.9 + \frac{0}{3} = 0.9 \text{ or } -0.9$$

- No minimum gap between $\beta_{(2)}$ and $\beta_{(3)}$
- Maximum gap between $\beta_{(2)}$ and $\beta_{(3)}$ is 0.9

Pairing $(\beta_{(1)}, \beta_{(3)})$

$$\gamma^{\text{min}}_{S_{\beta_{(1)},\beta_{(3)}}} = \pm d + \frac{\mathbb{I}^{\text{inclusive}}_S(\beta_g)}{G} = \pm 0.9 + \frac{3}{3} = 1.9 \text{ or } 0.1$$

$$\gamma^{\text{max}}_{S_{\beta_{(1)},\beta_{(3)}}} = \pm d + \frac{\mathbb{I}^{\text{exclusive}}_S(\beta_g)}{G} = \pm 0.9 + \frac{2}{3} = 1.57 \text{ or } -0.23$$

- Minimum gap between $\beta_{(1)}$ and $\beta_{(3)}$ is 0.1
- No maximum gap between $\beta_{(1)}$ and $\beta_{(3)}$

Mixed Logit (MXL) in Assortment Optimization

→ Heterogeneity (random) of customers

→ Correlation among alternatives

- β is a random parameter
- β follows any given continuous distribution
- $L_i(\beta)$ MNL for a given β (MNL kernel)

$$P_i^{\text{MXL}} = \int L_i(\beta) \cdot f(\beta) d\beta = \int L_i(\beta) dF(\beta) \tag{10}$$

Discrepancy (Probabilistic): Multidimensional Integral

Use the multidimensional integration to obtain the probability of draw $\beta_{(i)}$ falls within those ranges.

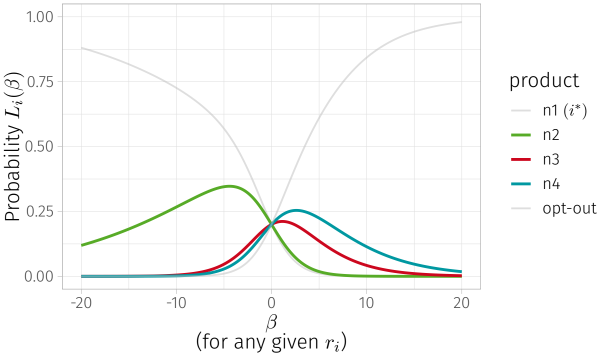
$$P(D_G \leq 0.9) = \int_0^{0.9} \int_{\beta_{(1)}}^{\min(\beta_{(1)}+0.9,1)} \int_{\max(\beta_{(1)}+0.1,\beta_{(2)})}^{\min(\beta_{(2)}+0.9,1)} 3!\, d\beta_{(3)}\, d\beta_{(2)}\, d\beta_{(1)}$$

Issue with Variation

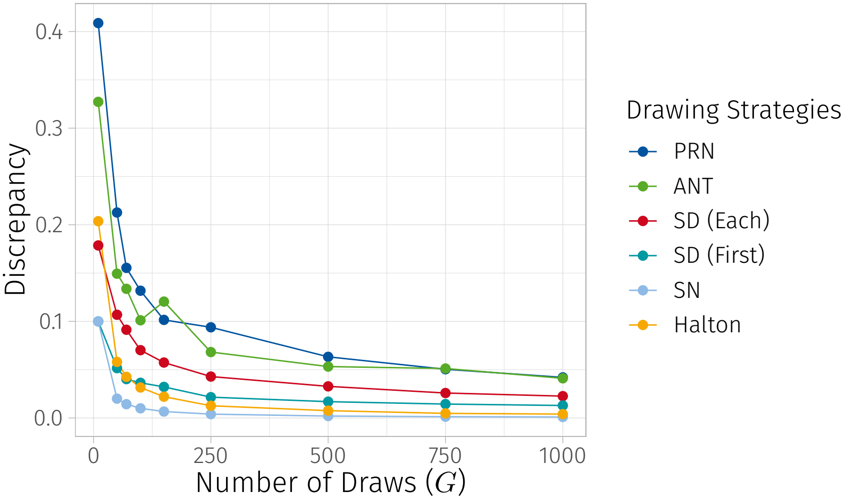
$$L_i(\beta) = \left(\frac{e^{-\beta r_i} Y_i}{1 + \sum_{j \in N} e^{-\beta r_j} Y_j} \right)$$

Issue:

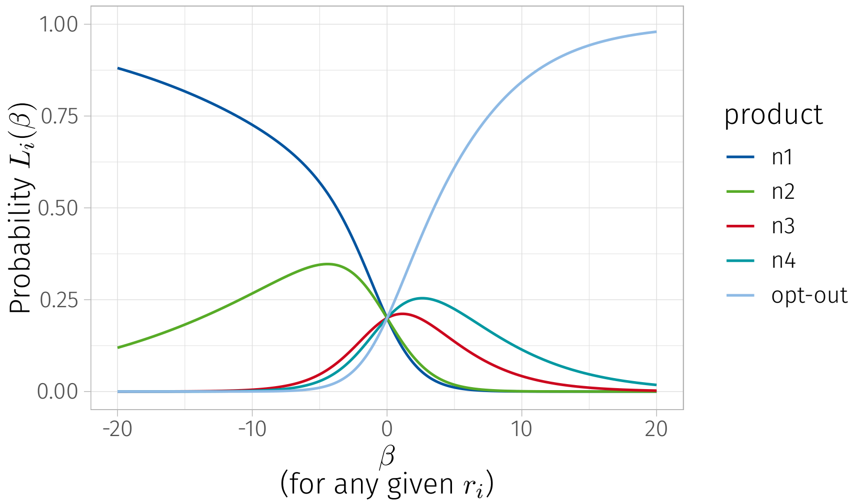
1. This might not be true when $n > 2$.
2. L_i is dependent on the assortment.



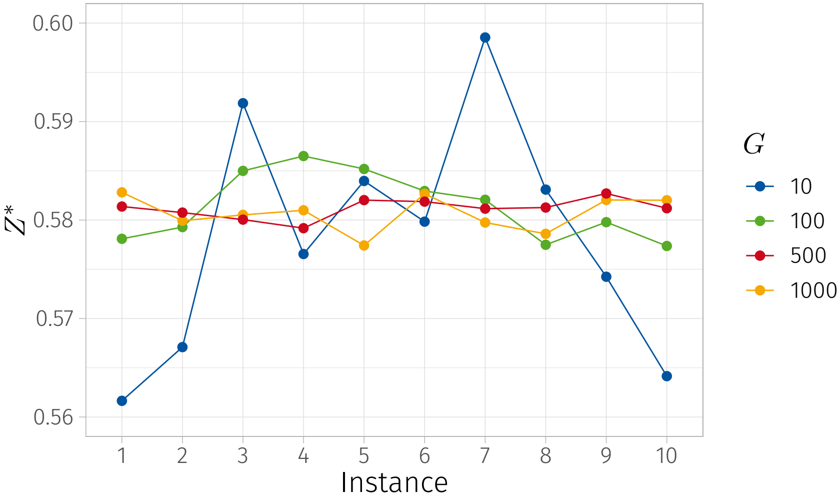
Discrepancy Comparison: All drawing strategies



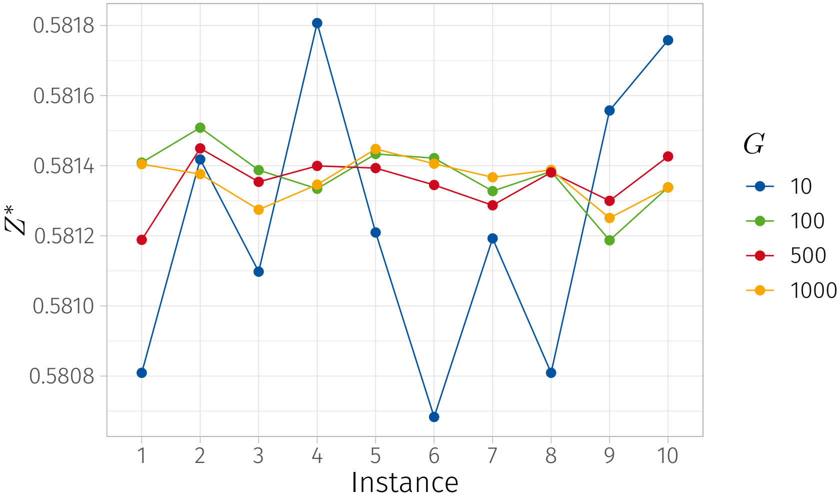
MXL Plot Enlarged



Z^* per Instance (PRN)



Z^* per Instance (ANT)



Z^* per Instance (Halton)